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ON THE RESTRICTED PROBLEM OF THREE BODIES

BY

J. F. STEFFENSEN

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i kommission hos Ejnar Munksgaard

Synopsis.

Three bodies are assumed to move in a plane, subject to Newton's law of gravitation, one of the bodies being infinitely small, and the two others moving in circles around their common centre of gravitation. To expand the coordinates of the small body in powers of the time is generally assumed to be impractical, but it is shown here that by introducing certain auxiliary dependent variables, the equations of motion are transformed into a differential system of the *second degree*, permitting to calculate the coefficients of the series by a set of recurrence formulas particularly adapted to the modern calculating machines. Sufficient conditions for the convergence of the resulting series are obtained, and a simple numerical example is given.

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1. We have in view the well-known particular case of the Problem of Three Bodies where the movement takes place in a plane, and two of the masses describe circles about their common centre of gravity, while the mass of the third body is infinitely small. Expansion of the coordinates in powers of the time t can be obtained by successive differentiations of the equations of motion, but this way of calculating the coefficients of the powers of t has been given up as too tedious[1]. We intend to show here that the calculation of the coefficients can be carried out with comparative ease when the equations of motion are transformed into a differential system of the *second degree*, permitting to calculate the coefficients of t^ν by a set of recurrence formulas, particularly adapted to the modern calculating machines. The process is closely related to that employed in one of my papers on the differential equations of G. W. Hill[2].

The equations of motion are given in Darwin's paper, p. 103. We write them, with a change of notation[3],

$$\left.\begin{aligned} &\frac{d^2p}{dt^2} - 2N\frac{dq}{dt} + Mp\,(r^{-3} - 1) + (p - 1)\,(s^{-3} - 1) = 0, \\ &\frac{d^2q}{dt^2} + 2N\frac{dp}{dt} + Mq(r^{-3} - 1) + q(s^{-3} - 1) = 0, \end{aligned}\right\} \qquad (1)$$

where

$$r^2 = p^2 + q^2, \qquad s^2 = r^2 + 1 - 2p, \qquad (2)$$

$$N^2 = M + 1. \qquad (3)$$

[1] G. H. Darwin: "Periodic Orbits". Acta mathematica, 21 (1897), 129—132.

[2] J. F. Steffensen: "On the Differential Equations of Hill in the Theory of the Motion of the Moon (II)". Acta mathematica, 95 (1956), 25—37.

[3] Darwin's x, y, n, ν, ϱ, C have in succession been replaced by p, q, N, M, s, K.

Jacobi's integral is

$$M\,(r^2+2\,r^{-1})+(s^2+2\,s^{-1})-\left(\frac{dp}{dt}\right)^2-\left(\frac{dq}{dt}\right)^2=K. \tag{4}$$

In these equations p and q are the coordinates of the infinitesimal body, the masses of the finite bodies are M and 1, their distance from each other 1, and the angular velocity of the system N.

Referring for further particulars to Darwin's paper we put

$$X=r^{-3}-1, \qquad Y=s^{-3}-1, \tag{5}$$

so that

$$\left.\begin{aligned} r\frac{dX}{dt}+3\,(X+1)\frac{dr}{dt}&=0,\\ s\frac{dY}{dt}+3\,(Y+1)\frac{ds}{dt}&=0.\end{aligned}\right\} \tag{6}$$

while (1) can be written

$$\left.\begin{aligned} \frac{d^2p}{dt^2}-2\,N\frac{dq}{dt}+MpX+pY-Y&=0,\\ \frac{d^2q}{dt^2}+2\,N\frac{dp}{dt}+MqX+qY&=0.\end{aligned}\right\} \tag{7}$$

For the determination of p, q, r, s, X, Y we have the 6 equations (7), (6) and (2) which we propose to satisfy by power series in t without making use of Jacobi's integral. We put

$$p=\sum_{\nu=0}^{\infty}a_\nu t^\nu, \qquad q=\sum_{\nu=0}^{\infty}b_\nu t^\nu, \tag{8}$$

$$r=\sum_{\nu=0}^{\infty}c_\nu t^\nu, \qquad s=\sum_{\nu=0}^{\infty}d_\nu t^\nu, \tag{9}$$

$$X=\sum_{\nu=0}^{\infty}e_\nu t^\nu, \qquad Y=\sum_{\nu=0}^{\infty}f_\nu t^\nu. \tag{10}$$

Inserting these series in the 6 equations and demanding that the coefficient of t^n shall vanish, we find by (7)

$$\left.\begin{aligned} &(n+1)(n+2)\,a_{n+2} - 2\,N(n+1)\,b_{n+1} + M\sum_{\nu=0}^{n} a_\nu e_{n-\nu} \\ &\qquad + \sum_{\nu=0}^{n} a_\nu f_{n-\nu} - f_n = 0, \\ &(n+1)(n+2)\,b_{n+2} + 2\,N(n+1)\,a_{n+1} + M\sum_{\nu=0}^{n} b_\nu e_{n-\nu} \\ &\qquad + \sum_{\nu=0}^{n} b_\nu f_{n-\nu} = 0, \end{aligned}\right\} \quad (11)$$

by (6)

$$\left.\begin{aligned} &\sum_{\nu=0}^{n} (\nu+1)\, e_{\nu+1} c_{n-\nu} + 3\sum_{\nu=0}^{n} (\nu+1)\, c_{\nu+1} e_{n-\nu} \\ &\qquad + 3\,(n+1)\, c_{n+1} = 0, \\ &\sum_{\nu=0}^{n} (\nu+1)\, f_{\nu+1} d_{n-\nu} + 3\sum_{\nu=0}^{n} (\nu+1)\, d_{\nu+1} f_{n-\nu} \\ &\qquad + 3\,(n+1)\, d_{n+1} = 0, \end{aligned}\right\} \quad (12)$$

and by (2)

$$\left.\begin{aligned} &\sum_{\nu=0}^{n} c_\nu c_{n-\nu} = \sum_{\nu=0}^{n} a_\nu a_{n-\nu} + \sum_{\nu=0}^{n} b_\nu b_{n-\nu}, \\ &\sum_{\nu=0}^{n} d_\nu d_{n-\nu} = \sum_{\nu=0}^{n} c_\nu c_{n-\nu} - 2\,a_n \qquad (n > 0), \\ &d_0^2 = c_0^2 + 1 - 2\,a_0. \end{aligned}\right\} \quad (13)$$

As initial values (constants of integration) we choose the coordinates and components of velocity of the infinitely small body at the time $t = 0$, that is a_0, a_1, b_0, b_1. Hence we find by (13), c_0 and d_0 being positive (since r and s represent distances from the finite masses)

$$c_0 = \sqrt{a_0^2 + b_0^2}, \qquad d_0 = \sqrt{c_0^2 + 1 - 2\,a_0}, \qquad (14)$$

whereafter by (10) and (5)

$$e_0 = c_0^{-3} - 1, \qquad f_0 = d_0^{-3} - 1. \qquad (15)$$

The remaining constants are calculated by the recurrence formulas (11)—(13). We state these in the form and order in which they are to be employed.

$$2\,c_0 c_n = \sum_{\nu=0}^{n} a_\nu a_{n-\nu} + \sum_{\nu=0}^{n} b_\nu b_{n-\nu} - \sum_{\nu=1}^{n-1} c_\nu c_{n-\nu}. \tag{16}$$

$$2\,d_0 d_n = \sum_{\nu=0}^{n} c_\nu c_{n-\nu} - \sum_{\nu=1}^{n-1} d_\nu d_{n-\nu} - 2\,a_n. \tag{17}$$

$$-\,n c_0 e_n = 3\sum_{\nu=1}^{n} \nu c_\nu e_{n-\nu} + \sum_{\nu=1}^{n-1} \nu e_\nu c_{n-\nu} + 3\,n c_n. \tag{18}$$

$$-\,n d_0 f_n = 3\sum_{\nu=1}^{n} \nu d_\nu f_{n-\nu} + \sum_{\nu=1}^{n-1} \nu f_\nu d_{n-\nu} + 3\,n d_n. \tag{19}$$

$$\left.\begin{aligned} -\,n\,(n+1)\,a_{n+1} = {} & M\sum_{\nu=0}^{n-1} a_\nu e_{n-\nu-1} + \sum_{\nu=0}^{n-1} a_\nu f_{n-\nu-1} \\ & -2\,N n b_n - f_{n-1}. \end{aligned}\right\} \tag{20}$$

$$-\,n\,(n+1)\,b_{n+1} = M\sum_{\nu=0}^{n-1} b_\nu e_{n-\nu-1} + \sum_{\nu=0}^{n-1} b_\nu f_{n-\nu-1} + 2\,N n a_n. \tag{21}$$

We give below the first few of these recurrence formulas.

$$\left.\begin{aligned} c_0 c_1 &= a_0 a_1 + b_0 b_1. \\ d_0 d_1 &= c_0 c_1 - a_1. \\ -\,c_0 e_1 &= 3\,c_1(e_0+1). \\ -\,d_0 f_1 &= 3\,d_1(f_0+1). \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} -\,2\,a_2 &= M a_0 e_0 + f_0(a_0-1) - 2\,N b_1, \\ -\,2\,b_2 &= M b_0 e_0 + f_0 b_0 + 2\,N a_1. \\ 2\,c_0 c_2 &= 2\,a_0 a_2 + a_1^2 + 2\,b_0 b_2 + b_1^2 - c_1^2. \\ 2\,d_0 d_2 &= 2\,c_0 c_2 + c_1^2 - d_1^2 - 2\,a_2. \\ -\,c_0 e_2 &= 2\,c_1 e_1 + 3\,c_2(e_0+1). \\ -\,d_0 f_2 &= 2\,d_1 f_1 + 3\,d_2(f_0+1). \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} -\,6\,a_3 &= M\,(a_0 e_1 + a_1 e_0) + f_0 a_1 + f_1(a_0-1) - 4\,N b_2. \\ -\,6\,b_3 &= M\,(b_0 e_1 + b_1 e_0) + b_0 f_1 + b_1 f_0 + 4\,N a_2. \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} c_0 c_3 &= a_0 a_3 + a_1 a_2 + b_0 b_3 + b_1 b_2 - c_1 c_2 . \\ d_0 d_3 &= c_0 c_3 + c_1 c_2 - d_1 d_2 - a_3 . \\ -3\, c_0 e_3 &= 5\, c_1 e_2 + 7\, c_2 e_1 + 9\, c_3 (e_0 + 1) . \\ -3\, d_0 f_3 &= 5\, d_1 f_2 + 7\, d_2 f_1 + 9\, d_3 (f_0 + 1) . \end{aligned}\right\} \tag{24}$$

It is seen that these forms lend themselves easily to the calculating machine.

2. In order to examine the convergence we write (16)—(21) in the following form where the constants of integration and those of zero order have been isolated. In (25), (29) and (30) we assume $n \geqslant 3$, in (26)—(28) $n \geqslant 2$.

$$\left.\begin{aligned} c_0 c_n = a_0 a_n + a_1 a_{n-1} + b_0 b_n + b_1 b_{n-1} - c_1 c_{n-1} \\ + \frac{1}{2} \sum_{\nu=2}^{n-2} (a_\nu a_{n-\nu} + b_\nu b_{n-\nu} - c_\nu c_{n-\nu}) . \end{aligned}\right\} \tag{25}$$

$$d_0 d_n = c_0 c_n - a_n + \frac{1}{2} \sum_{\nu=1}^{n-1} (c_\nu c_{n-\nu} - d_\nu d_{n-\nu}) . \tag{26}$$

$$-n c_0 e_n = 3\, n c_n (e_0 + 1) + 2 \sum_{\nu=1}^{n-1} \nu c_\nu e_{n-\nu} + n \sum_{\nu=1}^{n-1} c_\nu e_{n-\nu} . \tag{27}$$

$$-n d_0 f_n = 3\, n d_n (f_0 + 1) + 2 \sum_{\nu=1}^{n-1} \nu d_\nu f_{n-\nu} + n \sum_{\nu=1}^{n-1} d_\nu f_{n-\nu} . \tag{28}$$

$$\left.\begin{aligned} -n(n+1)\, a_{n+1} = a_0 (f_{n-1} + M e_{n-1}) + a_1 (f_{n-2} + M e_{n-2}) + \\ \sum_{\nu=2}^{n-2} a_\nu (f_{n-\nu-1} + M e_{n-\nu-1}) + a_{n-1} (f_0 + M e_0) - f_{n-1} - 2\, N n b_n . \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} -n(n+1)\, b_{n+1} = b_0 (f_{n-1} + M e_{n-1}) + b_1 (f_{n-2} + M e_{n-2}) \\ + \sum_{\nu=2}^{n-2} b_\nu (f_{n-\nu-1} + M e_{n-\nu-1}) + b_{n-1} (f_0 + M e_0) + 2\, N n a_n . \end{aligned}\right\} \tag{30}$$

We now put, as in an earlier paper[1],

$$K_\nu = \frac{\lambda^\nu}{\nu(\nu+1)} \qquad (\lambda > 0) \tag{31}$$

[1] Acta mathematica, 93 (1955), 173.

and assume that it has been proved for $2 \leqslant \nu \leqslant n$ that

$$| a_\nu | \leqslant AK_\nu, \qquad | b_\nu | \leqslant BK_\nu \tag{32}$$

and for $1 \leqslant \nu \leqslant n-1$ that

$$| c_\nu | \leqslant CK_\nu, \quad | d_\nu | \leqslant DK_\nu, \quad | e_\nu | \leqslant EK_\nu, \quad | f_\nu | \leqslant FK_\nu . \tag{33}$$

We then find sufficient conditions, by (25) — (28) for the validity of (33) in the case $\nu = n$, and by (29) — (30) for (32) in the case $\nu = n+1$, so that the inequalities (32) and (33) are valid for all ν under consideration. We proceed as in the paper quoted, making use of the identity

$$\left. \begin{aligned} K_\nu K_{m-\nu} = \lambda^m \Bigg[\left(\frac{1}{\nu} + \frac{1}{m-\nu} \right) \frac{1}{m(m+1)} - \\ \left(\frac{1}{\nu+1} + \frac{1}{m-\nu+1} \right) \frac{1}{(m+1)(m+2)} \Bigg] . \end{aligned} \right\} \tag{34}$$

From this, writing for abbreviation

$$s_n = \sum_{\nu=1}^{n} \frac{1}{\nu} \tag{35}$$

we obtain the sums[1]

$$\sum_{\nu=1}^{n-1} K_\nu K_{n-\nu} = 2 \frac{n-1+2\,s_{n-1}}{n(n+1)(n+2)} \lambda^n, \tag{36}$$

$$\sum_{\nu=2}^{n-2} K_\nu K_{n-\nu} = \left(2 \frac{n-1+2\,s_{n-1}}{(n+1)(n+2)} - \frac{1}{n-1} \right) \frac{\lambda^n}{n}, \tag{37}$$

$$\sum_{\nu=2}^{n-2} K_\nu K_{n-\nu-1} = \left(2 \frac{n-2+2\,s_{n-2}}{n(n+1)} - \frac{1}{2(n-2)} \right) \frac{\lambda^{n-1}}{n-1}, \tag{38}$$

$$\sum_{\nu=1}^{n-1} \nu K_\nu K_{n-\nu} = \frac{n-1+2\,s_{n-1}}{(n+1)(n+2)} \lambda^n . \tag{39}$$

[1] Interpreted as zero, if the upper limit of summation is less than the lower.

3. Dealing first with (25), we obtain, c_0 being positive, by (32) and (33)

$$\left.\begin{aligned} c_0 \mid c_n \mid \leqslant & (A \mid a_0 \mid + B \mid b_0 \mid) K_n \\ & + (A \mid a_1 \mid + B \mid b_1 \mid + C \mid c_1 \mid) K_{n-1} \\ & + \frac{1}{2}(A^2 + B^2 + C^2) \sum_{\nu=2}^{n-2} K_\nu K_{n-\nu} \end{aligned}\right\}$$

or

$$\left.\begin{aligned} c_0 \mid c_n \mid \leqslant & (A \mid a_0 \mid + B \mid b_0 \mid) \frac{\lambda^n}{n(n+1)} \\ & + (A \mid a_1 \mid + B \mid b_1 \mid + C \mid c_1 \mid) \frac{\lambda^{n-1}}{(n-1)n} \\ & + \frac{1}{2}(A^2 + B^2 + C^2)\left(2 \frac{n-1+2 s_{n-1}}{(n+1)(n+2)} - \frac{1}{n-1}\right)\frac{\lambda^n}{n}. \end{aligned}\right\} \tag{40}$$

If, now, we demand that the right-hand side of this inequality shall be $\leqslant c_0 C K_n = c_0 C \dfrac{\lambda^n}{n(n+1)}$, we obtain after multiplication by $n\,(n+1)\,\lambda^{-n}$ as a sufficient condition for the validity of $|c_\nu| \leqslant CK_\nu$ in all cases under consideration

$$\left.\begin{aligned} & A \mid a_0 \mid + B \mid b_0 \mid + (A \mid a_1 \mid + B \mid b_1 \mid + C \mid c_1 \mid) \frac{n+1}{n-1} \cdot \frac{1}{\lambda} \\ & + \frac{1}{2}(A^2 + B^2 + C^2)\left(2 \frac{n-1+2 s_{n-1}}{n+2} - \frac{n+1}{n-1}\right) \leqslant C c_0 . \end{aligned}\right\} \tag{41}$$

We replace this condition by a simpler but more rigid condition obtained by replacing the factors depending on n by absolute numbers which are at least as large.

Since $\dfrac{n+1}{n-1} = 1 + \dfrac{2}{n-1}$, this factor is constantly decreasing and may for $n \geqslant 2$ be replaced by 3.

Putting next

$$S_n = 2 \frac{n-1+2 s_{n-1}}{n+2} - \frac{n+1}{n-1}, \tag{42}$$

and observing that

$$s_n \leqslant 1 + \frac{1}{2} + \frac{1}{3} + \frac{n-3}{4} = \frac{13}{12} + \frac{n}{4} \tag{43}$$

so that

$$s_{n-1} \leqslant \frac{5}{6} + \frac{n}{4}$$

or $4\, s_{n-1} \leqslant n + \frac{10}{3}$, we find by inserting this in (42)

$$S_n \leqslant 2 - \frac{14}{3\,n+6} - \frac{2}{n-1}.$$

Hence

$$S_n < 2. \tag{44}$$

Observing finally that $\frac{n+1}{n-1} \leqslant 3$ for $n \geqslant 2$, we may replace (41) by the more rigid condition

$$\left.\begin{aligned} A\,|\,a_0\,| + B\,|\,b_0\,| + (A\,|\,a_1\,| + B\,|\,b_1\,| + C\,|\,c_1\,|)\frac{3}{\lambda} \\ + A^2 + B^2 + C^2 \leqslant c_0 C \end{aligned}\right\} \tag{45}$$

which is independent of n.

4. Next, as regards (26), we find by (36), corresponding to (41), the sufficient condition

$$Cc_0 + A + (C^2 + D^2)\frac{n - 1 + 2\,s_{n-1}}{n+2} \leqslant Dd_0. \tag{46}$$

The condition that the factor depending on n shall be steadily decreasing may be written in the form

$$s_{n-1} > \frac{5}{2} + \frac{2}{n} \tag{47}$$

which is satisfied for $n \geqslant 10$. We therefore have in this region[1]

$$\frac{n - 1 + 2\,s_{n-1}}{n+2} < \frac{3}{4} + \frac{1}{6}s_9 < \frac{5}{4} \tag{48}$$

[1] A table of s_n is found in S. Spitzer: Tabellen für die Zinses-Zinsen und Renten-Rechnung, Wien 1897, 369—370.

which is also valid for $n < 10$. We may therefore replace (46) by the simpler but more rigid condition

$$C c_0 + A + \frac{5}{4}(C^2 + D^2) \leqslant D d_0 . \tag{49}$$

5. From (27) we obtain by (39) and (36) as a sufficient condition

$$3 \mid e_0 + 1 \mid C + 4\, CE \frac{n - 1 + 2\, s_{n-1}}{n+2} \leqslant E c_0 , \tag{50}$$

and from this, by (48), the more rigid sufficient condition

$$3 \mid e_0 + 1 \mid C + 5\, CE \leqslant E c_0 . \tag{51}$$

Since (28) is obtained from (27) by a simple exchange of letters we may at once by (51) write down the following sufficient condition, resulting from (28)

$$3 \mid f_0 + 1 \mid D + 5\, DF \leqslant F d_0 . \tag{52}$$

6. As regards (29), we have, by (31)—(33) and (38),

$$\left.\begin{aligned} n\,(n+1) \mid a_{n+1} \mid \leqslant \mid a_0 \mid (F + ME) \frac{\lambda^{n-1}}{(n-1)\, n} \\ + \mid a_1 \mid (F + ME) \frac{\lambda^{n-2}}{(n-2)\,(n-1)} \\ + A\,(F + ME) \left(2 \frac{n - 2 + 2\, s_{n-2}}{n\,(n+1)} - \frac{1}{2\,(n-2)} \right) \frac{\lambda^{n-1}}{n-1} \\ + A \mid f_0 + M e_0 \mid \frac{\lambda^{n-1}}{(n-1)\, n} + F \frac{\lambda^{n-1}}{(n-1)\, n} + 2\, NB \frac{\lambda^{n}}{n+1} . \end{aligned}\right\} \tag{53}$$

If we demand that the right-hand side of this shall be $\leqslant n\,(n+1)\, A K_{n+1} = A \frac{n}{n+2} \lambda^{n+1}$, we obtain after multiplication by $\frac{n+2}{n} \lambda^{1-n}$ the condition

$$\left.\begin{aligned} &[|a_0|(F+ME)+A|f_0+Me_0|+F]\frac{n+2}{(n-1)n^2} \\ &+2NB\frac{n+2}{n(n+1)}\lambda+|a_1|(F+ME)\frac{n+2}{n(n-1)(n-2)}\frac{1}{\lambda} \\ &+A(F+ME)R_n \leqslant A\lambda^2 \end{aligned}\right\} \quad (54)$$

where

$$R_n = \frac{2(n+2)}{n(n-1)}\left(\frac{n-2+2s_{n-2}}{n(n+1)} - \frac{1}{4(n-2)}\right). \quad (55)$$

We proceed to show that

$$R_n \leqslant \frac{1}{8} \qquad (n \geqslant 3). \quad (56)$$

We write (55) in the form

$$R_n = \left(\frac{2}{n-1} + \frac{4}{n(n-1)}\right)\left(2\frac{s_{n-2}-1}{n(n+1)} + \frac{3}{4}\frac{n-3}{(n+1)(n-2)}\right)$$

where we may assume $n \geqslant 5$, since $R_3 = 0$, $R_4 = \frac{1}{8}$. Now the first factor in R_n is evidently decreasing, and the second factor is the sum of two decreasing expressions, since

$$\frac{s_{n-2}-1}{n(n+1)} > \frac{s_{n-1}-1}{(n+1)(n+2)} \qquad (n \geqslant 5)$$

which can be written

$$2(s_{n-2}-1) > 1 + \frac{1}{n-1},$$

and

$$\frac{n-3}{(n+1)(n-2)} > \frac{n-2}{(n+2)(n-1)} \qquad (n \geqslant 5)$$

which can be written

$$n(n-5)+2 > 0.$$

The remaining factors depending on n in (54) are steadily decreasing, and we find for $n \geqslant 3$

$$\frac{n+2}{(n-1)n^2} = \frac{1}{(n-1)n} + \frac{2}{(n-1)n^2} \leqslant \frac{5}{18}, \tag{57}$$

$$\frac{n+2}{n(n+1)} = \frac{1}{n+1} + \frac{2}{n(n+1)} \leqslant \frac{5}{12}, \tag{58}$$

$$\left.\begin{aligned} \frac{n+2}{n(n-1)(n-2)} &= \frac{1}{(n-1)(n-2)} \\ &+ \frac{2}{n(n-1)(n-2)} \leqslant \frac{5}{6}. \end{aligned}\right\} \tag{59}$$

Inserting finally the limits (56) — (59) in (54), we obtain the more rigid, but of n independent, sufficient condition

$$\left.\begin{aligned} &\frac{5}{18}\left[|a_0|(F+ME) + A|f_0 + Me_0| + F\right] + \frac{5}{6}NB\lambda \\ &+ \frac{5}{6}|a_1|(F+ME)\frac{1}{\lambda} + \frac{1}{8}A(F+ME) \leqslant A\lambda^2. \end{aligned}\right\} \tag{60}$$

7. As regards finally (30), a comparison with (29) shows that we obtain the same form as (53), the only difference being that a and b, A and B have been exchanged and the term $F\dfrac{\lambda^{n-1}}{(n-1)n}$ left out. We may therefore immediately write down the sufficient condition corresponding to (60)

$$\left.\begin{aligned} &\frac{5}{18}\left[|b_0|(F+ME) + B|f_0 + Me_0|\right] + \frac{5}{6}NA\lambda \\ &+ \frac{5}{6}|b_1|(F+ME)\frac{1}{\lambda} + \frac{1}{8}B(F+ME) \leqslant B\lambda^2. \end{aligned}\right\} \tag{61}$$

8. The result of the preceding investigation is that, if for a certain $n \geqslant 3$ it has been proved that (32) is satisfied for $2 \leqslant \nu \leqslant n$ and (33) for $1 \leqslant \nu \leqslant n-1$, and if, besides, the inequalities

(45), (49), (51), (52), (60) and (61) are all satisfied, then the expansions (8) — (10) are convergent, provided that $\Sigma K_\nu \mid t \mid^\nu$ converges, that is, for $\mid t \mid \leqslant \frac{1}{\lambda}$.

The question arises whether, when the constants of integration are arbitrarily given, it is always possible to find such values of λ, A, B, C, D, E, F that the six inequalities are all satisfied. We proceed to prove that this is really so.

To begin with, λ can always be chosen so large that (60) and (61) are satisfied, no matter what values the other constants possess, and (45) can for sufficiently large λ be replaced by

$$A \mid a_0 \mid + B \mid b_0 \mid + \frac{5}{4}(A^2 + B^2) < C\left(c_0 - \frac{5}{4}C\right) \tag{62}$$

while the three remaining inequalities which we write in the form

$$A + C\left(c_0 + \frac{5}{4}C\right) \leqslant D\left(d_0 - \frac{5}{4}D\right), \tag{63}$$

$$3 \mid e_0 + 1 \mid C \leqslant E\,(c_0 - 5\,C), \tag{64}$$

$$3 \mid f_0 + 1 \mid D \leqslant F\,(d_0 - 5\,D), \tag{65}$$

are unchanged. Now it follows from (64) and (65) that we must choose

$$C < \frac{1}{5}c_0, \qquad D < \frac{1}{5}d_0, \tag{66}$$

after which (64) and (65) are satisfied, provided that we choose E and F sufficiently large. After this, (62) will be satisfied, if we choose A and B sufficiently small in comparison with C, and (63) if A and C are sufficiently small in comparison with D. In thus choosing small values for A, B, C and D we do not run into difficulties, because (31) — (33) show that small values of these constants can be compensated by choosing λ sufficiently large.

There is, thus, always a solution for sufficiently small values of $\mid t \mid$, if $c_0 > 0$, $d_0 > 0$ as assumed in (14).

9. If at the time $t = 0$ we have $q = 0$, $\frac{dp}{dt} = 0$, that is $b_0 = 0$, $a_1 = 0$, certain simplifications occur. In that case there are only the two arbitrary constants a_0 and b_1 left, and we find first by (14) and (15), if $a_0 \neq 0$ and $a_0 \neq 1$,

$$\left.\begin{aligned} c_0 = | a_0 |, \quad d_0 = | a_0 - 1 |, \quad e_0 = | a_0 |^{-3} - 1, \\ f_0 = | a_0 - 1 |^{-3} - 1. \end{aligned}\right\} \quad (67)$$

The recurrence formulas now show that b_ν vanishes when ν is an even number, and the other coefficients when ν is odd. Under these circumstances the working formulas (16)—(21) are best written thus

$$\left.\begin{aligned} -(2n-1)\,2na_{2n} = M\sum_{\nu=0}^{n-1} a_{2\nu}e_{2n-2\nu-2} + \sum_{\nu=0}^{n-1} a_{2\nu}f_{2n-2\nu-2} \\ -2N(2n-1)\,b_{2n-1} - f_{2n-2}. \end{aligned}\right\} \quad (68)$$

$$\left.\begin{aligned} 2c_0c_{2n} = \sum_{\nu=0}^{n} a_{2\nu}a_{2n-2\nu} + \sum_{\nu=1}^{n} b_{2\nu-1}b_{2n-2\nu+1} \\ -\sum_{\nu=1}^{n-1} c_{2\nu}c_{2n-2\nu}. \end{aligned}\right\} \quad (69)$$

$$2d_0d_{2n} = \sum_{\nu=0}^{n} c_{2\nu}c_{2n-2\nu} - \sum_{\nu=1}^{n-1} d_{2\nu}d_{2n-2\nu} - 2a_{2n}. \quad (70)$$

$$-nc_0e_{2n} = 3\sum_{\nu=1}^{n} \nu c_{2\nu}e_{2n-2\nu} + \sum_{\nu=1}^{n-1} \nu e_{2\nu}c_{2n-2\nu} + 3nc_{2n}. \quad (71)$$

$$-nd_0f_{2n} = 3\sum_{\nu=1}^{n} \nu d_{2\nu}f_{2n-2\nu} + \sum_{\nu=1}^{n-1} \nu f_{2\nu}d_{2n-2\nu} + 3nd_{2n}. \quad (72)$$

$$\left.\begin{aligned} -2n(2n+1)\,b_{2n+1} = M\sum_{\nu=1}^{n} b_{2\nu-1}e_{2n-2\nu} \\ + \sum_{\nu=1}^{n} b_{2\nu-1}f_{2n-2\nu} + 4Nna_{2n}. \end{aligned}\right\} \quad (73)$$

The first few of these formulas are

$$\left.\begin{aligned} -2\,a_2 &= Ma_0e_0 + f_0(a_0 - 1) - 2\,Nb_1. \\ c_0c_2 &= a_0a_2 + \frac{1}{2}b_1^2. \\ d_0d_2 &= c_0c_2 - a_2. \\ -c_0e_2 &= 3\,c_2(e_0+1). \\ -d_0f_2 &= 3\,d_2(f_0+1). \end{aligned}\right\} \quad (74)$$

$$-6\,b_3 = Mb_1e_0 + b_1f_0 + 4\,Na_2. \quad (75)$$

$$\left.\begin{aligned} -12\,a_4 &= M(a_0e_2 + a_2e_0) + f_2(a_0-1) + a_2f_0 - 6\,Nb_3. \\ 2\,c_0c_4 &= 2\,a_0a_4 + a_2^2 + 2\,b_1b_3 - c_2^2. \\ 2\,d_0d_4 &= 2\,c_0c_4 + c_2^2 - d_2^2 - 2\,a_4. \\ -c_0e_4 &= 2\,c_2e_2 + 3\,c_4(e_0+1). \\ -d_0f_4 &= 2\,d_2f_2 + 3\,d_4(f_0+1). \end{aligned}\right\} \quad (76)$$

$$-20\,b_5 = M(b_1e_2 + b_3e_0) + b_1f_2 + b_3f_0 + 8\,Na_4. \quad (77)$$

$$\left.\begin{aligned} -30\,a_6 &= M(a_0e_4 + a_2e_2 + a_4e_0) + f_4(a_0-1) + a_2f_2 \\ &\quad + a_4f_0 - 10\,Nb_5. \\ c_0c_6 &= a_0a_6 + a_2a_4 + b_1b_5 + \frac{1}{2}b_3^2 - c_2c_4. \\ d_0d_6 &= c_0c_6 + c_2c_4 - d_2d_4 - a_6. \\ -3\,c_0e_6 &= 3(c_2e_4 + 2\,c_4e_2 + 3\,c_6e_0) + e_2c_4 + 2\,e_4c_2 + 9\,c_6. \\ -3\,d_0f_6 &= 3\,(d_2f_4 + 2\,d_4f_2 + 3\,d_6f_0) + f_2d_4 + 2\,f_4d_2 + 9\,d_6. \end{aligned}\right\} \quad (78)$$

10. As a simple numerical example of the application of (74) — (78) we choose $a_0 = \frac{1}{2}$, $b_1 = -1$ besides the already assumed $b_0 = 0$, $a_1 = 0$ leading to (67). For N and M we choose the values $N = 1{\cdot}1$, $M = {\cdot}21$ which satisfy (3). The results are given in the table below.

ν	a_ν	c_ν	d_ν
0	·5	·5	·5
2	·2825	1·2825	·7175
4	—·4332729	—4·407273	—2·4107271
6	1·3130591	19·199425	8·728045

ν	e_ν	f_ν	ν	b_ν
0	7·	7·	1	—1·
2	—61·56	—34·44	3	1·2045
4	527·3519	214·5577	5	—2·687845
6	—4442·1231	—1319·5487		

A partial check on these calculations is obtained by calculating the value of Jacobi's constant K by (4) for various values of t. I have found

$$t = 0, \qquad K = 4{\cdot}1425$$

$$t = {\cdot}03, \qquad K = 4{\cdot}1424999$$

which seems satisfactory.

As regards the convergence, (32) and (33) are satisfied by the coefficients given in the table if, for instance, we choose $\lambda = 20$, $A = {\cdot}005$, $B = {\cdot}002$, $C = {\cdot}02$, $D = {\cdot}04$, $E = 1{\cdot}2$, $F = 3.2$, and since these values also satisfy all the six inequalities (45), (49), (51), (52), (60) and (61), the expansions (8) — (10) are at least convergent for $|\,t\,| \leqslant \frac{1}{20}$.

This space of time may at first appear to be small, but the expansion for q shows that it corresponds to a movement in the vertical direction of nearly one tenth of the original distance of the infinitesimal body from either of the two finite bodies.

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